Computing with lights

Thierry Monteil
CNRS – LIPN – Université Paris 13

Signal machines [Durand-Lose 2003]

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- continuous space ($\mathbb{R}$)

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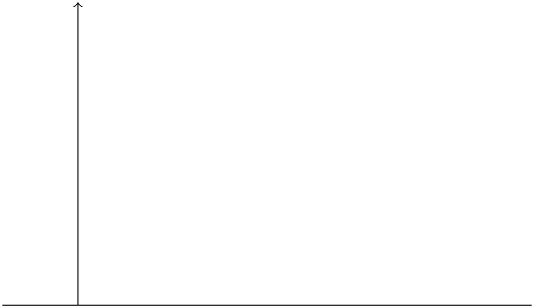
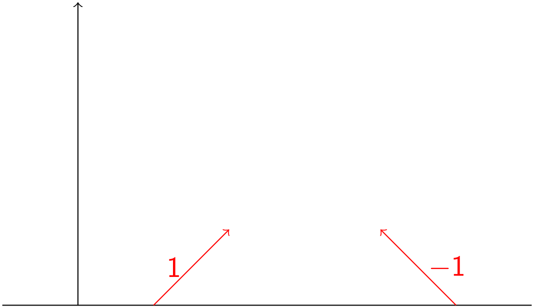
- continuous space ($\mathbb{R}$)
- continuous time ($\mathbb{R}_+$)



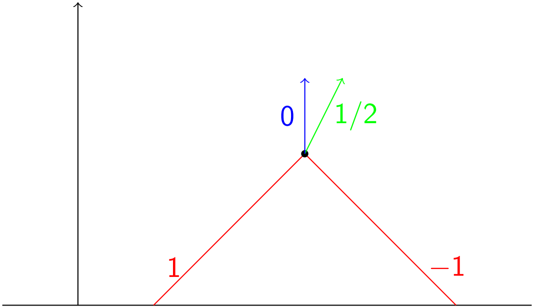
Signal machines [Durand-Lose 2003]

- continuous space ($\mathbb{R}$)
- continuous time ($\mathbb{R}_+$)
- finitely many signals with constant speed



Signal machines [Durand-Lose 2003]

- continuous space ($\mathbb{R}$)
- continuous time ($\mathbb{R}_+$)
- finitely many signals with constant speed
- discrete events (deterministic collision rules)



Example: Middle machine

Example: Middle machine

Colors: Black Green Red

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Meta-signals:

Example: Middle machine

Colors: Black Green Red

Meta-signals:

Collision rules:

Example: Middle machine

Colors: Black Green Red

Meta-signals:

Collision rules:

Initial configuration:

Example: Middle machine

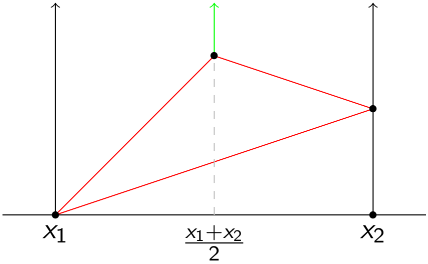
Colors: Black Green Red

Meta-signals:

Collision rules:

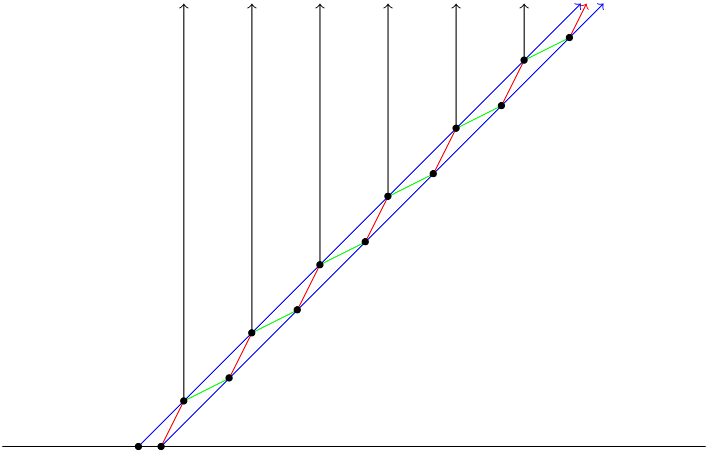
Initial configuration:

Execution:

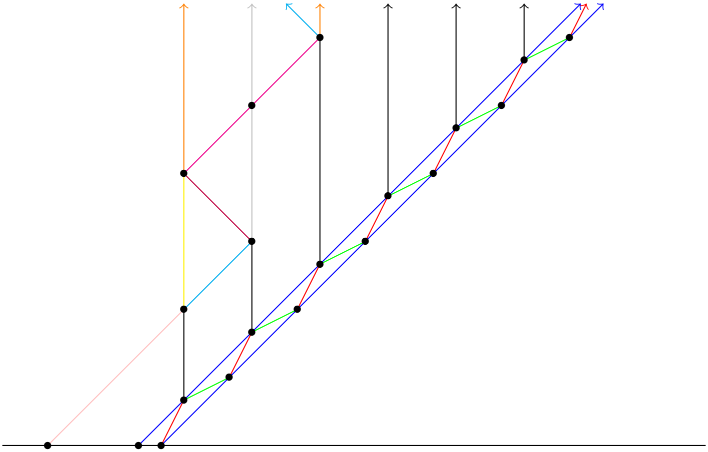


What can signal machines compute ?

Signal machines can construct cells



Signal machines can simulate Turing machines

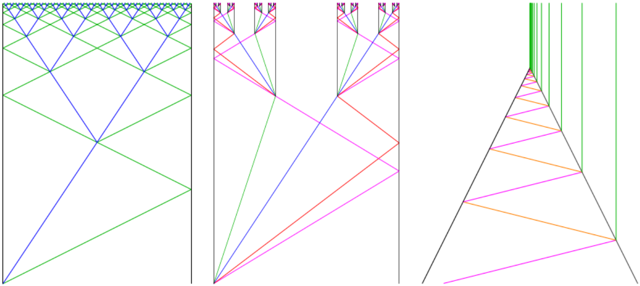


The model without accumulations is well understood

Theorem (Durand-Lose 2007)

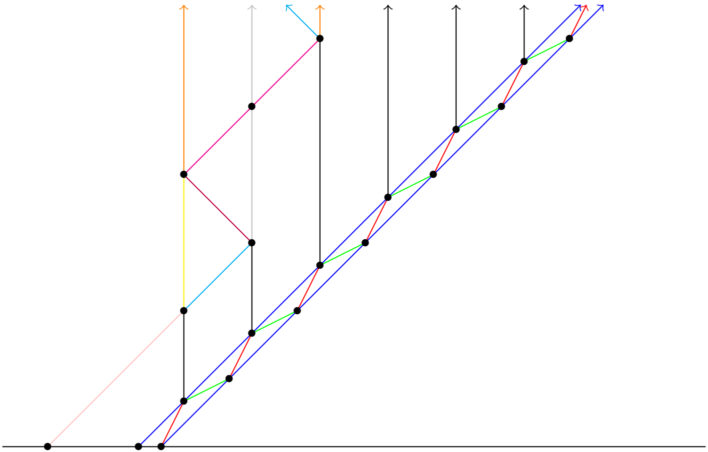
Signal machines without accumulations and linear-BSS model are computationally equivalent.

Acumulations can do weird things

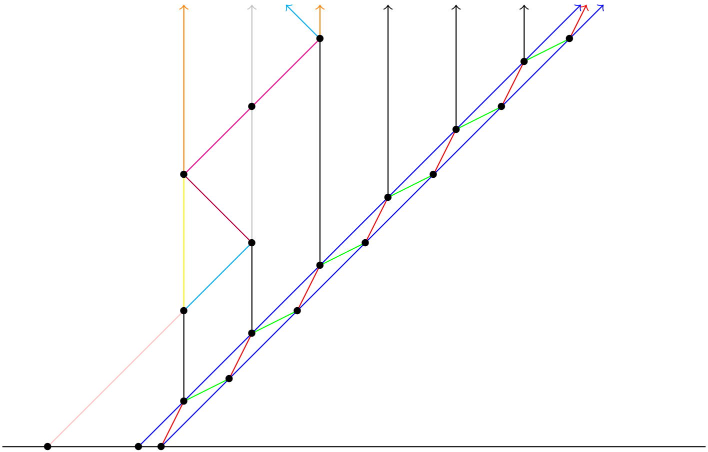


How frugal can interesting signal machines be ?

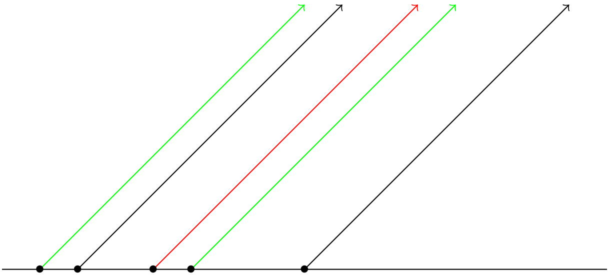
Four speeds were enough in the previous construction



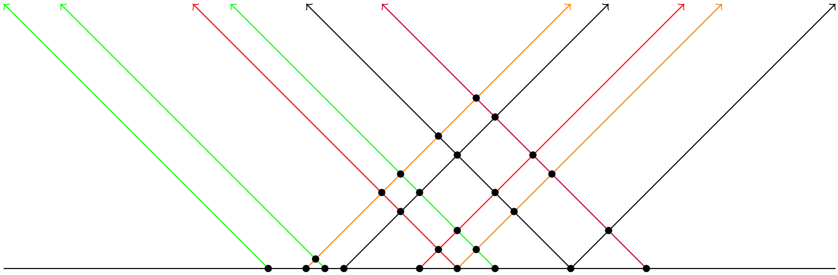
Four speeds were enough in the previous construction
Can we do less ?



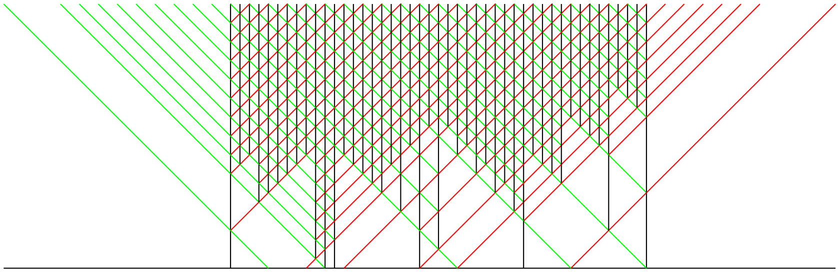
One speed: no collision



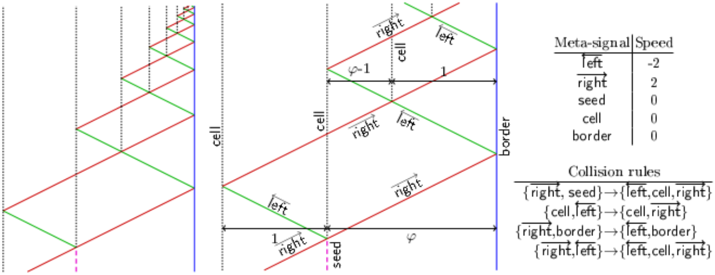
Two speeds: finitely many collisions



Three speeds + rational ratios: included in bounded mesh



Three speeds + one irrational ratio: Turing again [Durand-Lose CiE 2013]

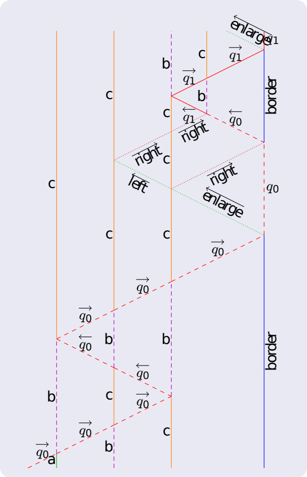


(a) Fractal (b) Fractal construction

Conjecture [Durand-Lose CiE 2013]

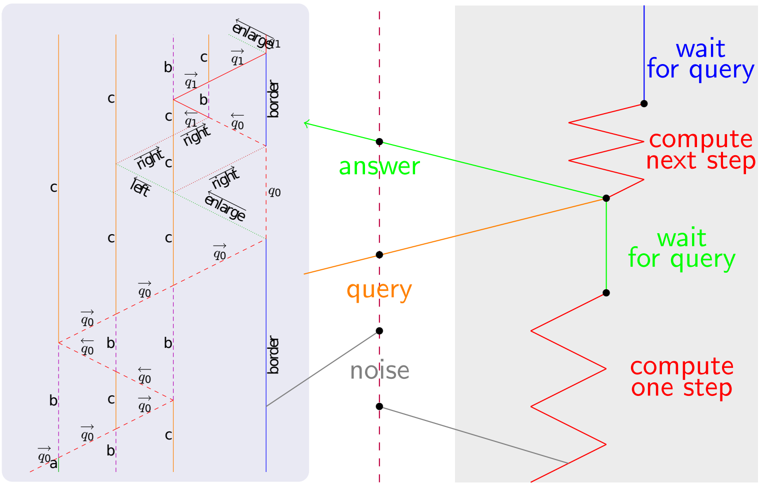
An irrational ratio is an important piece of information (it could encode the halting problem). We conjecture that it is possible to use it as an oracle.

Simulation of Turing machines



Turing machine
[Durand-Lose 2003]

Simulation of Turing machines with oracle



Turing machine
[Durand-Lose 2003]

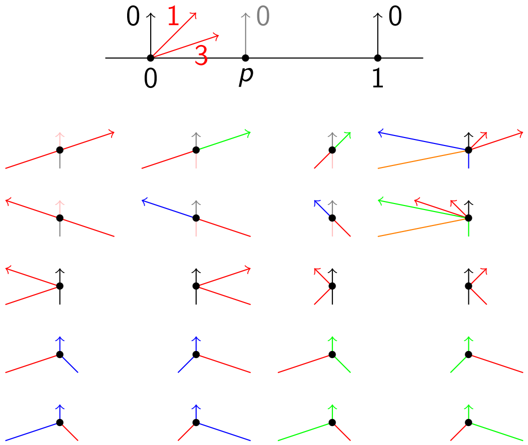
firewall

oracle
(to be designed)

Linear-BSS can easily extract binary expansion of real numbers

```
def binary_oracle(p):
    while True:
        if p < 1/2:
            yield 0
            p = 2*p
        elif 1/2 < p:
            yield 1
            p = 2*p-1
        else:
            return
```

Skip the general construction: iterate the middle construction as a dichotomy + interaction layer



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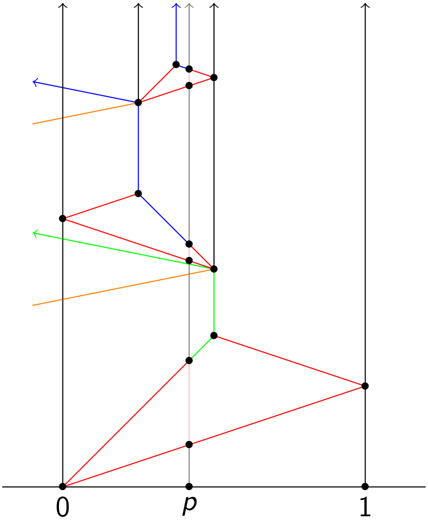


How frugal ?

The previous construction uses 5 speeds, and we can not hope to use less than 4 speeds to compute the half (resp. double) of any distance, otherwise we can create an accumulation (resp. an unbounded set) of collisions from a signal machine with at most 3 speeds and rational input.

Can we produce infinite expansion without multiplication by constants ?

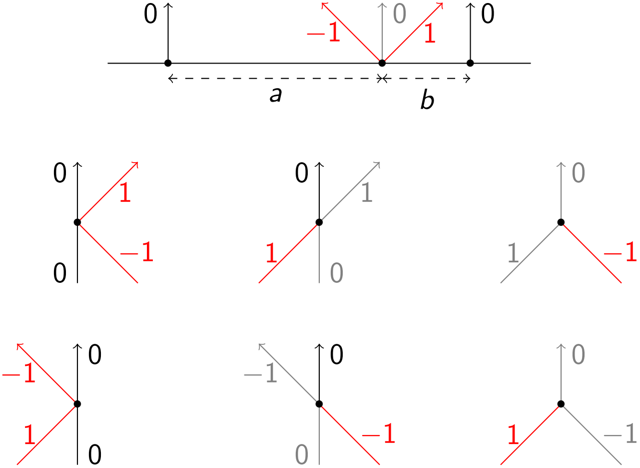
“addition-BSS” model will also create unboundedness

Could we think of a kind of “subtraction-BSS” restricted model ?

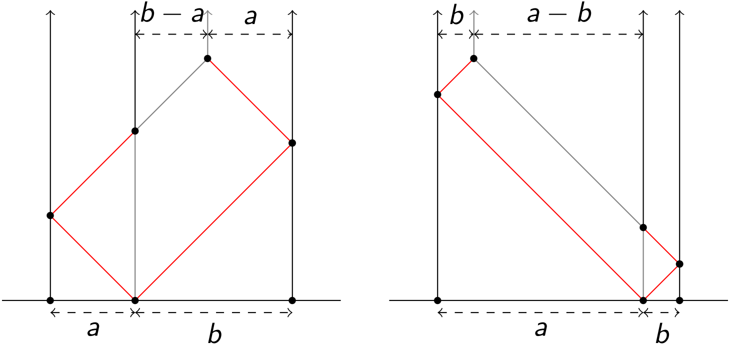
Continued fractions !

```
def continued_fractions_oracle(p):
a,b = p,1-p
while True:
    if a < b:
        yield 0
        a,b = b-a,a
    elif b < a:
        yield 1
        a,b = b,a-b
    else:
        return
```

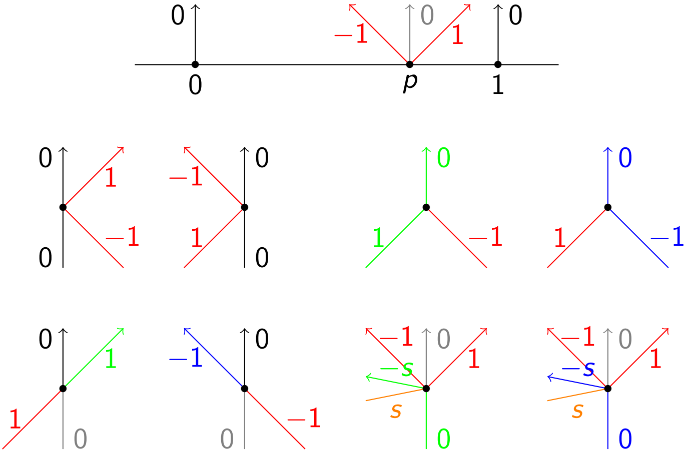
Basic block: a frugal subtraction machine



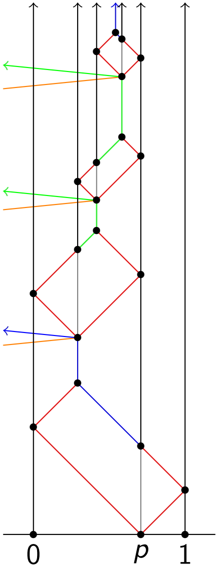
Basic block: a frugal subtraction machine



Continued fractions: iterate and interact



Continued fractions: iterate and interact



Continued fractions

Theorem

Every non-ultimately alternating infinite binary sequence appears for exactly one irrational number $p \in (0, 1)$.

Continued fractions

Theorem

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Every finite binary sequence appears for exactly one rational number $p \in (0, 1)$.

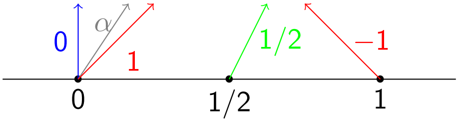
When α is a rational number, the continued fraction algorithm eventually goes to zero (corresponds to both billiards reaching the gray line simultaneously), and we have to define a new rule for this case (purple stands for "end of computation"):

Three speeds and one irrational parameter for the whole system

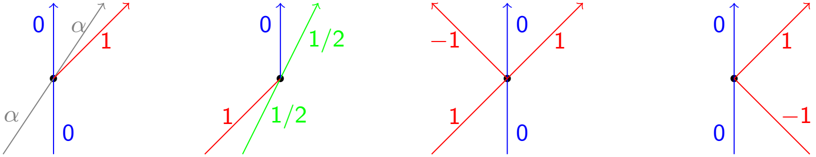
The Turing machine described in [Durand-Lose CiE 2013] also works with 3 speeds and a single irrational ratio. The speeds can be chosen identical for the Turing machine and the continued fraction oracle. However, for the Turing machine, the irrational parameter is the golden ratio, which allows to create cells accumulating on a single side. Here, the positions of the vertical strips which are created is not increasing, but oscillating in a way that depends on the continued fraction expansion of the irrational parameter. However, it is possible, still using only 3 speeds, to add some rules to the continued fraction machine that swap a and b before each subtraction step when $a > b$, so that each new vertical strips becomes larger than the previous one. In particular, a single irrational parameter can be first duplicated and used to construct both the oracle (as described in this paper) and the creation of cells for the Turing machine.

What computes this signal machine ($\alpha \in (0,1)$) ?

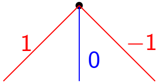
Initial configuration:



Transitions:



Accepting configuration:



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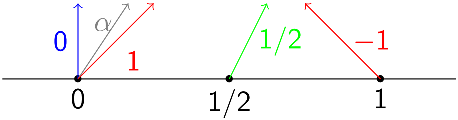
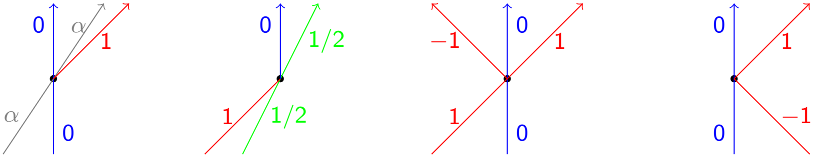
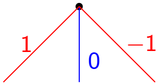


Transitions: [hint: Let $x = 1/\alpha - 1$]

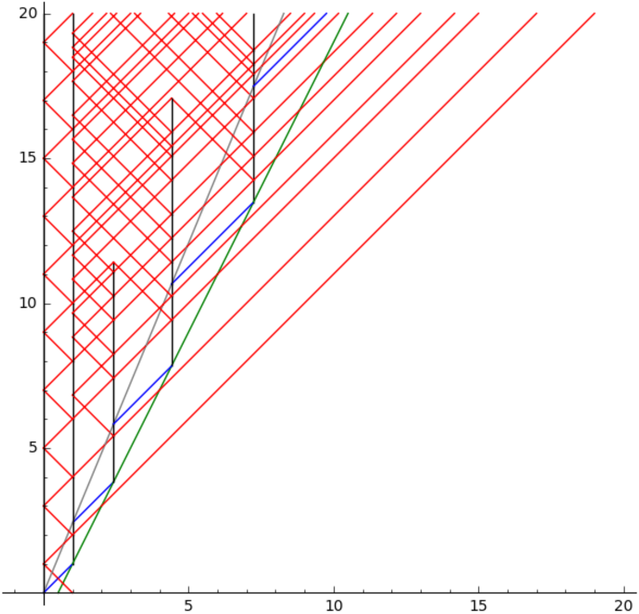


(builds and exponential frame) (parallel billiard (fork)) (don't go back)

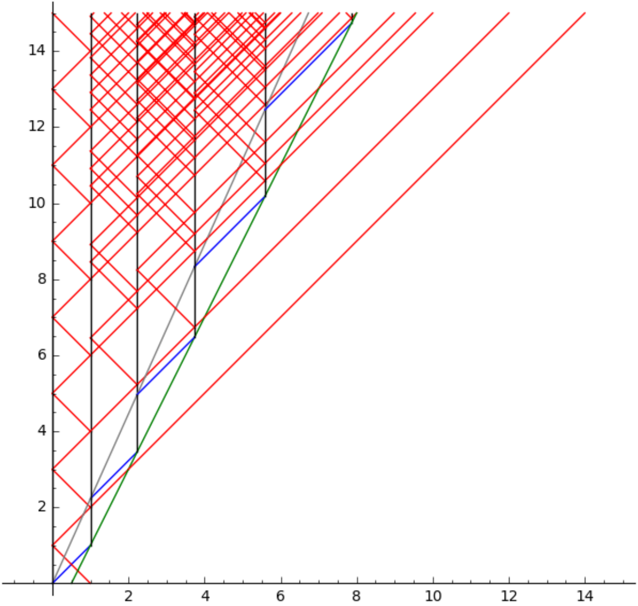
Accepting configuration:



It accepts $\sqrt{2}$



It does not accept $\pi/7$



Theorem

The previous machine reaches the accepting configuration if, and only if, α *is an algebraic number.*

A contradiction?

Theorem

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Theorem (Durand-Lose 2007)

Signal machines and linear-BSS model are computationally equivalent.

Theorem

No linear-BSS machine can semi-decide the algebraicity of real numbers.

A contradiction?

Theorem

The previous machine reaches the accepting configuration if, and only if, α is an algebraic number.

Theorem (Durand-Lose 2007)

Signal machines and linear-BSS model are computationally equivalent.

Theorem

No linear-BSS machine can semi-decide the algebraicity of real numbers.

The previous construction is cheating, since the parameter α is not an input (signal position), but is part of the machine's *definition* (signal speed). Hence, we have a defined an uncountable family of machines, not a single one. Don't panic.

Appendix

Theorem

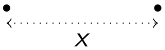
No linear-BSS machine can semi-decide the algebraicity of real numbers.

Proof.

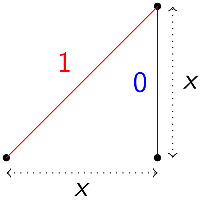
If M is such a machine, there are finitely many real numbers $(\alpha_1, \ldots, \alpha_n)$ such that every operation done by M are addition $(+)$, subtraction $(-)$, setting constant 0 or 1, multiply by some α_i, comparison $(\leq)$.
Let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, and let x be an algebraic number not in K.
Let T be the tree corresponding to all possible behaviours of M, and let P be the path in T corresponding to its behaviour on the input x. Since M accepts x, the path P if finite and the machine outputs 0 at the end of P.
Along the path P, all states of M are of the form $k_1 x + k_2$ with k_1 and k_2 in K, and each branching of T encountered along P can be reduced to a comparison of the form $k_1 x + k_2 \leq 0$ with $k_1 \neq 0$.

So the set S of inputs whose path in T is P is a finite intersection of sets defined by inequalities of the form: $k_1 x + k_2 \leq 0$ with $k_i \in K$ and $k_1 \neq 0$. Since x is not in K, x belongs to the interrior of each set ($k_1 x + k_2$ can not be equal to zero), hence the set S has non empty interior: there exists a transcendental number y (in S, whose path is P), that is accepted by M. □

Space-time equivalence



Space-time equivalence



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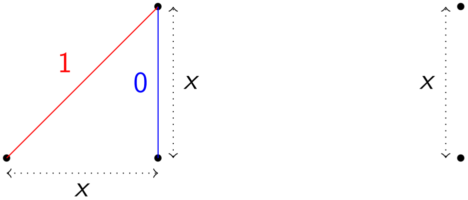

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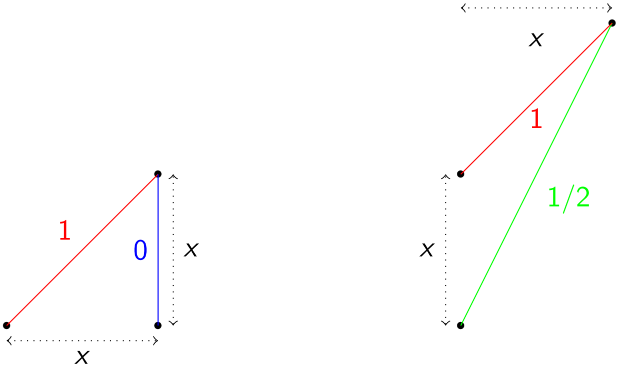

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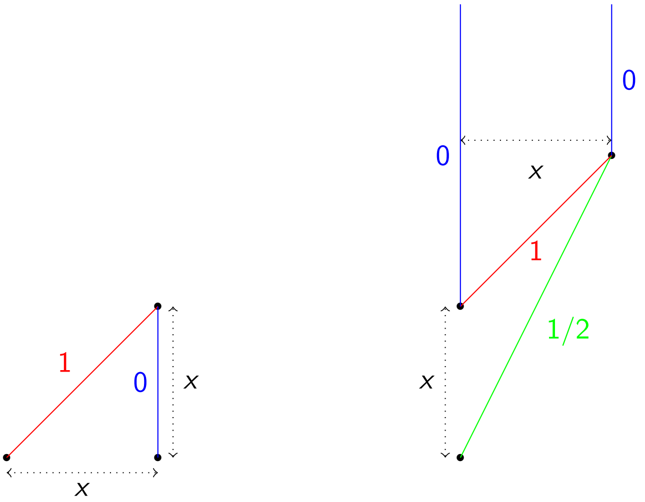
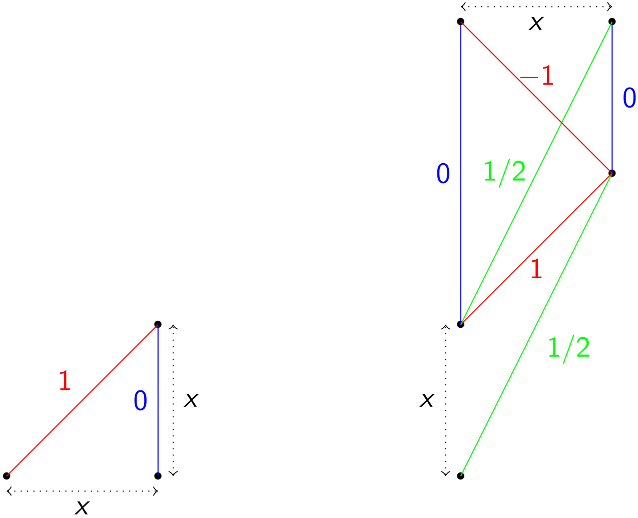
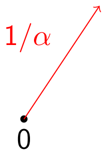


Space-time equivalence



Slope is stronger than space (or time)



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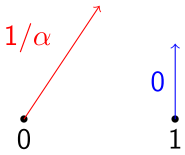

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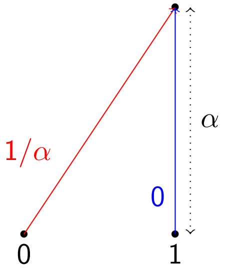
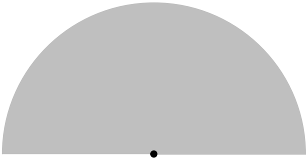


Explosions

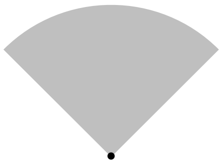
Let us introduce explosions as pencils of signals of all speeds.



Explosions

Let us introduce explosions as pencils of signals of all speeds.

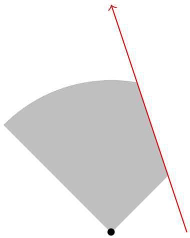
- bounded speed



Explosions

Let us introduce explosions as pencils of signals of all speeds.

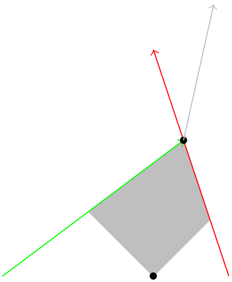
- bounded speed
- cutted by other signals



Explosions

Let us introduce explosions as pencils of signals of all speeds.

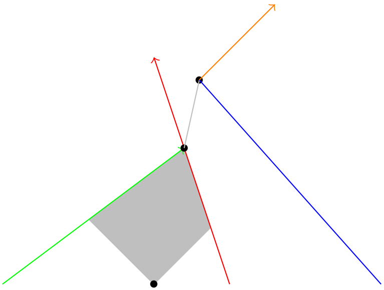
- bounded speed
- cutted by other signals
- can pass through events



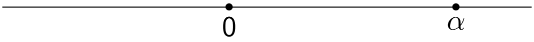
Explosions

Let us introduce explosions as pencils of signals of all speeds.

- bounded speed
- cutted by other signals
- can pass through events
- can interact with other signals



Explosions (slope-speed-space-time equivalence)



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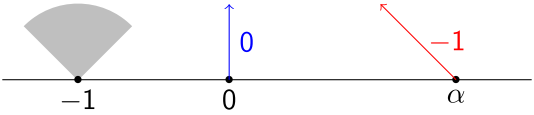

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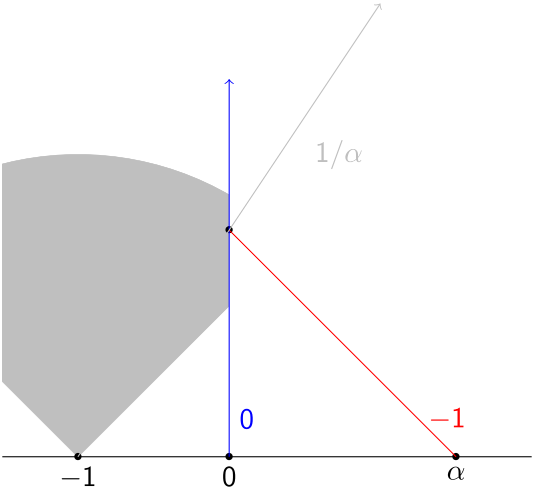
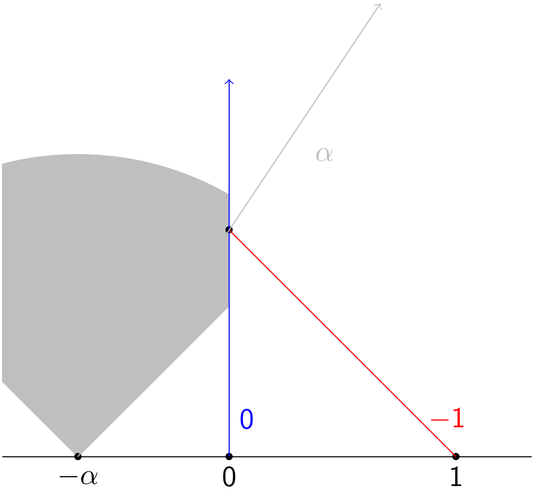


space is stronger than slope

Explosions (slope-speed-space-time equivalence)



space is stronger than speed

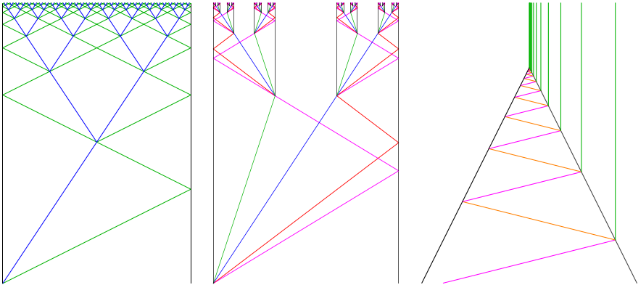
Explosions (products and divisions)

Theorem

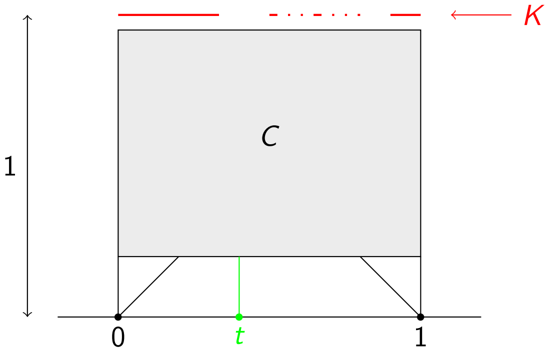
Explosive signal machines and full-BSS model are computationally equivalent.

Accumulations: a very (resp. too much) powerful model

Acumulations: can draw nice (self-similar) compacts

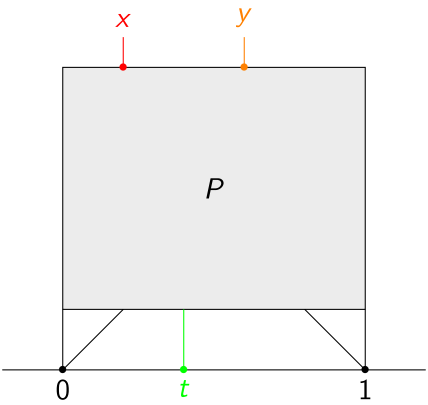


Accumulations: a universal compact drawer



$$\exists C,\ \forall K \subseteq [0,1] \text{ compact},\ \exists t \in [0,1],\ \mathrm{acc}_1(C(t)) = K$$

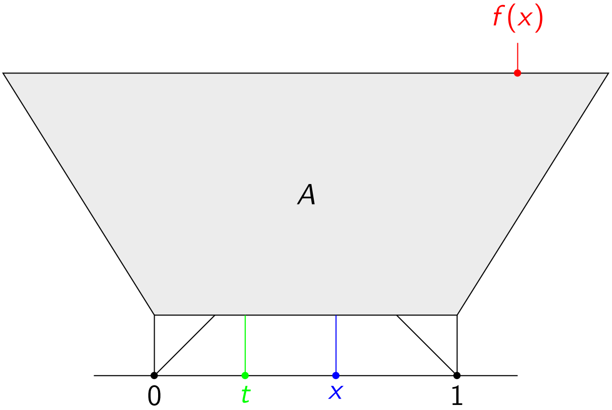
Accumulations: a Peano curve as a building block



$\exists P,\ t \mapsto P(t) = (x, y)$ is a continuous surjection $[0,1] \to [0,1]^2$

(note that the execution of the machine P itself is not continuous)

Accumulations: a universal analytic function



$$\exists A,\ \forall f \in C^{\omega}([0,1]),\ \exists t \in [0,1],\ \forall x \in [0,1],\ A(t,x) = f(x)$$